Strong-Coupling Behavior of ϕ^4-Theories and Critical Exponents

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We use the recently-developed variational perturbation theory to continue the renormalization constants of three-dimensional ϕ^4-theories to the regime of strong bare coupling g_0. In this limit, they are found to behave like powers in g_0, which determine all critical exponents. Convergent strong-coupling expansions are found for all renormalization group functions which converge well even in the weak-coupling limit.

I. INTRODUCTION

Some time ago, the Feynman-Kleinert variational approximation to path integrals [1] has been extended to a systematic variational perturbation expansion [2]. For an anharmonic oscillator, this expansion converges uniformly and exponentially fast, like $e^{-\text{const}\times N^{1/3}}$ in the order N of the approximation [3,4]. The uniformity of the convergence has permitted us to derive convergent strong-coupling expansions from divergent weak-coupling expansions [5,6]. The finite convergence radius g_s of the strong-coupling expansion turned out to govern the speed of convergence of the entire approach [7,8], the constant in the above exponential being directly related to g_s.

Since convergent strong-coupling expansions can be obtained so easily from divergent weak-coupling expansions, is was straight-forward to develop a simple algorithm for finding uniformly convergent optimal interpolations to functions for which one knows both several weak-coupling as well as strong-coupling expansion coefficients [9].

The purpose of this paper is to point out that the same algorithm can be used to calculate the behavior of the renormalization constants of three-dimensional ϕ^4-theories for *all* coupling strengths, from which we can extract the limit of infinite bare coupling constant g_0. Since g_0 always appears in the combination g_0/m, we thereby obtain the zero-mass limit of the theory, which describes the critical behavior of wide classes of many-body systems related to these field theories by universality of infrared behavior. Indeed, we find for the renormalization constants power behaviors in the bare coupling constant, which in Wilson's theory of critical behavior are derived only by an awkward ϵ-expansion in the dimension of the theory around the scale invariant dimension four.

II. REMINDER OF STRONG-COUPLING THEORY

Let us briefly recall the algorithm developed in [9], by which a divergent weak-coupling expansion of the type $E_N(g_0) = \sum_{n=0}^{N} a_n g_0^n$ is turned into a strong-coupling expansion $E_M(g_0) = g_0^{p/q} \sum_{m=0}^{M} b_m (g_0^{-2/q})^m$ with a finite convergence radius g_s. Examples treated in [9] were the anharmonic oscillator with parameters $p = 1/3,\ q = 3$ for the energy eigenvalues, and the Fröhlich polaron with $p = 1,\ q = 1$ for the ground-state energy and $p = 4,\ q = 1$ for the mass. For the mass of the polaron, the summation gave quite a different results from Feynman's, calling for further studies of this system.

As described in detail in [3], the first step is to rewrite the weak-coupling expansion with the help of an auxiliary scale parameter κ as

$$E_N(g_0) = \kappa^p \sum_{n=0}^{N} a_n \left(\frac{g_0}{\kappa^q}\right)^n \tag{1}$$

where κ is eventually set equal to 1. We shall see below that the quotient p/q parametrizes the *leading power behavior* in g_0 of the strong-coupling expansion, whereas $2/q$ characterizes the *approach* to the leading power behavior. In a second step we replace κ by the identical expression

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$$\kappa \to \sqrt{K^2 + \kappa^2 - K^2} \tag{2}$$

containing a dummy scaling parameter K. The series (1) is then reexpanded in powers of g_0 up to the order N, thereby treating $\kappa^2 - K^2$ as a quantity of order g_0. The result is most conveniently expressed in terms of dimensionless parameters $\hat{g}_0 \equiv g_0/K^q$ and $\sigma(K) equiv (1-\hat{\kappa}^2)/\hat{g}_0$, where $\hat{\kappa} \equiv \kappa/K$, and we have suppressed g_0 and κ in the arguments of $\sigma(K)$. Then the replacement (2) amounts to

$$\kappa \longrightarrow K(1-\sigma\hat{g}_0)^{1/2}, \tag{3}$$

so that the reexpanded series reads explicitly

$$W_N(g_0, K) = K^p \sum_{n=0}^{N} \varepsilon_n(\sigma(K))\,(\hat{g}_0)^n\,, \tag{4}$$

with the coefficients:

$$\varepsilon_n(\sigma) = \sum_{j=0}^{n} a_j \begin{pmatrix} (p-qj)/2 \\ n-j \end{pmatrix} (-\sigma)^{n-j}. \tag{5}$$

For any fixed g_0 and κ, we form the first and second derivatives of $W_N(g_0,K)$ with respect to K, calculate the K-values of the extrema and the turning points, and select the smallest of these as the optimal scaling parameter K_N. The function $W_N(g_0) \equiv W_N(g_0, K_N)$ constitutes the Nth variational approximation $E_N(g_0)$ to the function $E(g_0)$.

It is easy to take this approximation to the strong-coupling limit $g_0 \to \infty$. For this we observe that (4) has the scaling form

$$W_N(g_0, K) = K^p w_N(\hat{g}_0, \hat{\kappa}^2). \tag{6}$$

For dimensional reasons, the optimal K_N increases with g_0 like $K_N \approx g_0^{1/q} c_N$, so that $\hat{g}_0$ becomes asymptotically constant, say $\hat{g}_0 \to c_L^{-q}$, implying that remain finite in the strong-coupling limit $\sigma \to 1/\hat{g}_0 = c_L^q$. The dimensionless $\hat{\kappa}^2$ tends to zero like $1/[c_L(g_0/\kappa^q)^{1/q}]^2$. Hence

$$W_N(g_0, K_N) \approx g_0^{p/q} c_N^p w_N(c_N^{-q}, 0). \tag{7}$$

In this limiting form, c_N plays the role of the variational parameter to be determined by the optimal extremum or turning point of $c_N^p w_N(c_N^{-q}, 0)$.

The full strong-coupling expansion is obtained by expanding $w_N(\hat{g}_0, \hat{\kappa}^2)$ in powers of $\hat{\kappa}^2 = (g_0/\kappa^q \hat{g}_0)^{-2/q}$. The result is

$$W_N(g_0) = g_0^{p/q}\left[b_0(\hat{g}_0) + b_1(\hat{g}_0)\left(\frac{g_0}{\kappa^q}\right)^{-2/q} + b_2(\hat{g}_0)\left(\frac{g_0}{\kappa^q}\right)^{-4/q} + \ldots\right] \tag{8}$$

with

$$b_n(\hat{g}_0) = \frac{1}{n!} w_N^{(n)}(\hat{g}_0, 0)\hat{g}_0^{(2n-p)/q}\,, \tag{9}$$

where $w_N^{(n)}(\hat{g}_0, \hat{\kappa}^2)$ is the nth derivatives of $w_N(\hat{g}_0, \hat{\kappa}^2)$ with respect to $\hat{\kappa}^2$. Explicitly:

$$\frac{1}{n!} w_N^{(n)}(\hat{g}_0, 0) = \sum_{l=0}^{N} (-1)^{l+n} \sum_{j=0}^{l-n} a_j \begin{pmatrix} (p-qj)/2 \\ l-j \end{pmatrix} \begin{pmatrix} l-j \\ n \end{pmatrix} (-\hat{g}_0)^j. \tag{10}$$

The optimal expansion of the energy (8) is obtained by expanding

$$\hat{g}_0 = c_N^{-q}\left[1 + \gamma_1\left(\frac{g_0}{\kappa^q}\right)^{-2/q} + \gamma_2\left(\frac{g_0}{\kappa^q}\right)^{-4/q} + \ldots\right],$$

and finding the optimal extremum (or turning point) in the resulting polynomials of $\gamma_1, \gamma_2, \ldots$. In this way we obtain a systematic strong-coupling expansion in powers of $(g_0/\kappa^q)^{-2/q}$

$$W_L(g_B) = g_B^{p/q}\left[\bar{b}_0 + \bar{b}_1\left(\frac{g_B}{\kappa^q}\right)^{-2/q} + \bar{b}_2\left(\frac{g_B}{\kappa^q}\right)^{-4/q} + \dots\right]. \tag{11}$$

In practice, the coefficients $\bar{b}_0$ are determined successively as follows: First we optimize $b_0(\hat{g}_B)$ at $\hat{g}_B = c_N^{-q}$ yielding $\bar{b}_0 = b_0(c_N^{-q})$. At the same value of $\hat{g}_B$ we calculate $\bar{b}_1 = b_1(c_N^{-q})$, and further coefficients $b_i(c_N^{-q})$, $i = 2, 3, \dots$, and their derivatives $b_i'(c_N^{-q})$, $b_i''(c_N^{-q}) \dots$. From these we determine the remaining optimized coefficients $\bar{b}_i$ in the strong-coupling expansion (11) by combining $b_i(c_N^{-q})$ as specified in Table I. This procedure will be applied to quantum field theory in Section VI.

III. CONVERGENCE

The available number of weak-coupling coefficients is limited to $N = 6$ for three-dimensional ϕ^4-theories [10–12], and to $N = 5$ for theories in $4 - \epsilon$ dimensions [13]. In either case it will be important to know the specific way in which the approximations $W_N(\hat{g}_0)$ approach the correct result $W_\infty(\hat{g}_0)$. This will permit us to extrapolate the results found for $N = 1,\ 2,\ \dots, 6$ to $N \to \infty$ with some reliability. To do this, we must adapt the discussion for the harmonic oscillator in Ref. [7] to the general case.

We remove the factor κ^p from the function $E(g_0)$, defining the reduced quantity $\bar{E}(\bar{g}_0) = E(g_0)/\kappa^p$ as a function of the reduced coupling constant $\bar{g}_0 \equiv g_0/\kappa^q$. We further assume the strong-coupling growth $g_0^{p/q}$ of the function $E(g_0)$ to be less than linear, so that $\bar{E}(\bar{g}_0)$ satisfies a once-subtracted dispersion relation

$$\bar{E}(\bar{g}_0) = a_0 + \frac{\bar{g}_0}{2\pi i}\int_0^{-\infty}\frac{d\bar{g}_0'}{\bar{g}_0'}\frac{\operatorname{disc}\bar{E}(\bar{g}_0')}{\bar{g}_0' - \bar{g}_0}, \tag{12}$$

where $\operatorname{disc}\bar{E}(\bar{g}_0)$ is the discontinuity across the left-hand cut in the complex g_0-plane (from below to above). An expansion of the integrand in powers of $\bar{g}_0$ up to $\bar{g}_0^N$ reproduces of course the initial perturbation series (1), where the expansion coefficients are moment integrals over the discontinuity:

$$a_k = \frac{1}{2\pi i}\int_0^{-\infty}\frac{d\bar{g}_0}{\bar{g}_0^{k+1}}\operatorname{disc}\bar{E}(\bar{g}_0). \tag{13}$$

The dispersion relation (12) can also be used to derive moment integrals for the reexpansion coefficient $\varepsilon_l(\sigma)$ in (4), (5). For the dimensionless coupling constant $\bar{g}_0$, the replacement (3) becomes

$$\bar{g}_0 \longrightarrow \tilde{g}_0(\hat{g}_0) \equiv \frac{\hat{g}_0}{(1 - \sigma\hat{g}_0)^{q/2}}. \tag{14}$$

Because of the prefactor κ^p in (1), the replacement (2) also produces a prefactor $K^p/\kappa^p = (1 - \sigma\hat{g}_0)^{p/2}$ to the function $E(g_0)$. For the reduced function $\hat{E}(\hat{g}_0) \equiv E(g_0)/K^p$, which depends only on the reduced coupling constant $\hat{g}_0$, we thus obtain a dispersion relation

$$\hat{E}(\hat{g}_0) = (1 - \sigma\hat{g}_0)^{p/2}\left[a_0 + \frac{\tilde{g}_0(\hat{g}_0)}{2\pi i}\int_0^{-\infty}\frac{d\bar{g}_0'}{\bar{g}_0'}\frac{\operatorname{disc}\bar{E}(\bar{g}_0')}{\bar{g}_0' - \tilde{g}_0(\hat{g}_0)}\right]. \tag{15}$$

This function satisfies a dispersion relation in the complex $\hat{g}_0$-plane. If C denotes the image of the left-hand cut in original dispersion relation (12), as it arises from the mapping (14), and if $\operatorname{disc}_C E(\hat{g}_0)$ denotes the discontinuity across this cut, that dispersion relation reads

$$\hat{E}(\hat{g}_0) = a_0 + \frac{\hat{g}_0}{2\pi i}\int_C\frac{d\hat{g}_0'}{\hat{g}_0'}\frac{\operatorname{disc}_C\hat{E}(\hat{g}_0')}{\hat{g}_0' - \hat{g}_0}. \tag{16}$$

An expansion of the integrand in powers of $\hat{g}_0$ yields moment integrals for the desired reexpansion coefficients $\varepsilon_k(\sigma)$:

$$\varepsilon_k(\sigma) = \frac{1}{2\pi i}\int_C\frac{d\hat{g}_0}{\hat{g}_0^{k+1}}\operatorname{disc}_C\hat{E}(\hat{g}_0). \tag{17}$$

The discontinuity in these integrals can be derived from the dispersion relation (15). The function $\tilde{g}_0(\hat{g}_0)$ in (14) carries the left-hand cut in the complex $\bar{g}_0$-plane over into several cuts in the $\hat{g}_0$-plane. In Fig. 1 we show the image cuts arising for $q = 3$, where $\omega = 2/3$ (the case of the anharmonic oscillator discussed in [7]). They run along the contours C_1, $C_{\bar{1}}$, C_2 ,$C_{\bar{2}}$, C_3, the last being caused by the powers $q/2$ and $p/2$ of $1-\sigma\hat{g}_0$ in the mapping (14) and the prefactor in (15), respectively. Let $\bar{D}(\bar{g}_0)$ abbreviate the reduced discontinuity in the original dispersion relation (12):

$$\bar{D}(\bar{g}_0) \equiv \mathrm{disc}\bar{E}(\bar{g}_0) = 2i\mathrm{Im}\,\bar{E}(\bar{g}_0 - i\eta), \qquad \bar{g}_0 \leq 0. \tag{18}$$

Then the discontinuities across the various cuts are

$$\operatorname*{disc}_{C_{1,\bar{1},2,\bar{2}}} \hat{E}(\hat{g}_0) = (1-\sigma\hat{g}_0)^{p/2}\bar{D}(\hat{g}_0(1-\sigma\hat{g}_0)^{-q/2}), \tag{19}$$

$$\operatorname*{disc}_{C_3} \hat{E}(\hat{g}_0) = -2i(\sigma\hat{g}_0 - 1)^{p/2} \times \left[a_0 - \int_0^\infty \frac{d\bar{g}_0'}{2\pi}\frac{\hat{g}_0(\sigma\hat{g}_0-1)^{-q/2}}{\bar{g}_0'^2 + \hat{g}_0^2(\sigma\hat{g}_0-1)^{-q}}\bar{D}(-\bar{g}_0')\right]. \tag{20}$$

For small negative $\bar{g}_0$, the discontinuity is given by a standard semiclassical approximation with the typical form [14]:

$$\bar{D}(\bar{g}_0) \approx -i \text{ const} \times \bar{g}_0^b\, e^{a/\bar{g}_0}. \tag{21}$$

The exponential plays the role of a Boltzmann factor for the activation of a classical solution to the field equations, wheras the prefactor accounts for the entropy of the field fluctuations around this solution.

Let us denote by $\varepsilon_k(C_i)$ the contributions of the different cuts to the integral (17). After inserting (21) into (19), we obtain from the cut along C_1 the semiclassical approximation

$$\varepsilon_k(C_1) \approx \text{const} \times \int_{C_1}\frac{d\hat{g}_0}{2\pi}\frac{1}{\hat{g}_0^{k+1}}(1-\sigma\hat{g}_0)^{p/2-bq/2}\hat{g}_0^b\, e^{a(1-\sigma\hat{g}_0)^{q/2}/\hat{g}_0}. \tag{22}$$

For the kth term of the series $S_k \equiv \varepsilon_k\hat{g}_0^k$, this yields a large-$k$ estimate

$$S_k \propto \left[\int_{C_\gamma}\frac{d\gamma}{2\pi}e^{f_k(\gamma)}\right]\gamma^k, \tag{23}$$

where $\gamma \equiv \sigma\hat{g}_0$, and $f_k(\gamma)$ is the function

$$f_k(\gamma) \approx -k\log(-\gamma) + \frac{a\sigma}{\gamma}(1-\gamma)^{q/2}\frac{a\sigma}{\gamma}(1-\gamma)^{q/2} + (b-1)\log(-\gamma) + \dots\ .. \tag{24}$$

For large k, the saddle point approximation yields via the extremum at $\gamma \underset{k\to\infty}{\longrightarrow} \gamma_k = -a\sigma/k$:

$$f_k \underset{k\to\infty}{\longrightarrow} k\log(k/ea\sigma) - aq\sigma/2 + (b-1)\log\frac{a\sigma}{k} + \dots\ . \tag{25}$$

The constant $-aq\sigma/2$ in this limiting expression arises when expanding the second term of Eq. (24) into a Taylor series, $(a\sigma/\gamma)(1-\gamma)^{q/2} = a\sigma/\gamma_k - aq\sigma/2 + \dots$. Only the first two terms in (25) contribute to the large-k limit. Thus, to leading order in k, the kth term of the reexpanded series becomes

$$S_k \propto e^{-aq\sigma/2}\left(\frac{-3k}{e}\right)^k\left(\frac{\hat{g}}{4}\right)^k. \tag{26}$$

The corresponding reexpansion coefficients

$$\varepsilon_k \propto e^{-aq\sigma/2}E_k \tag{27}$$

have the remarkable property of growing in precisely the same manner with k as the initial expansion coefficients E_k, except for an overall suppression factor $e^{-aq\sigma/2}$, as discussed in Ref. [3].

To estimate the convergence of the variational perturbation expansion (4), we note that $\sigma\hat{g} = 1-\hat{\kappa}^2$ is for large K smaller than unity, so that the powers $(\sigma\hat{g})^k$ by themselves would yield a convergent series. An optimal reexpansion of the reduced function $\hat{E}(\hat{g}_0)$ can be achieved by choosing, for a given large maximal order N of the expansion, a parameter σ proportional to N:

$$\sigma \approx \sigma_N \equiv cN. \tag{28}$$

Inserting this into (24), we obtain for large $k = N$

$$f_N(\gamma) \approx N \left[-\log(-\gamma) + \frac{ac}{\gamma}(1-\gamma)^{q/2} \right]. \tag{29}$$

The extremum of this function lies at

$$1 + \frac{ac}{\gamma}(1-\gamma)^{q/2-1} \left[1 + \left(\frac{q}{2} - 1 \right) \gamma \right] = 0. \tag{30}$$

The constant c may be chosen in such a way that the large exponent proportional to N in the exponential function $e^{f_N(\gamma)}$ due to the first term in (29) is canceled by an equally large contribution from the second term, i.e., we require at the extremum

$$f_N(\gamma) = 0. \tag{31}$$

The two equations (30) and (31) are solved by certain constant values of $\gamma < 0$ and c. In contrast to the extremal γ of Eq. (24) which dominates the large-k limit, the extremal γ in the present limit, in which k is also large but of the same size as N, remains finite (the previous estimate held for $k \gg N$). Accordingly, the second term $(ac/\gamma)(1-\gamma)^{q/2}$ in $f_N(\gamma)$ contributes in full, not merely via the first two Taylor expansion terms of $(1-\gamma)^{q/2}$, as it did in (25).

Since $f_N(\gamma)$ vanishes at the extremum, the Nth term in the reexpansion has the order of magnitude

$$S_N(C_1) \propto (\sigma_N \hat{g}_0)^N = \left(1 - \frac{1}{K_N^2} \right)^N. \tag{32}$$

According to (28), the scale parameter K_N grows for large N like

$$K_N \sim \sigma_N^{1/q} g_0^{1/q} \sim (cNg_0)^{1/3}. \tag{33}$$

As a consequence, the last term of the series decreases for large N like

$$S_N(C_1) \propto \left[1 - \frac{1}{(\sigma_N g_0)^{2/q}} \right]^N \approx e^{-N/(\sigma_N g_0)^{2/q}} \approx e^{-N^{1-2/q}/(cg_0)^{2/q}}. \tag{34}$$

This estimate does not yet explain the exponentially fast convergence of the variational perturbation expansion in the strong-coupling limit, observed in [6]. For the contribution of the cut C_1 to S_N, the derivation of such a behavior requires including the approach of σ_N to the large-N behavior (28), which has the same general form as the strong-coupling expansion (8),

$$\sigma_N \sim cN \left(1 + \frac{c'}{N^{2/q}} + \dots \right), \tag{35}$$

as it turns out with positive c'. By inserting this σ_N into $f_N(\gamma)$ of (29), we find an extra exponential factor which dominates the large-N behavior for at infinite coupling $\hat{g}_0$:

$$e^{\Delta f_N} \approx \exp \left[-N \log(-\gamma) \frac{c'}{N^{2/q}} \right] \approx e^{-c'' N^{1-2/q}}. \tag{36}$$

What about the contributions of the other cuts? For $C_{\bar{1}}$, the integral in (17) runs from $\hat{g} = -2/\sigma$ to $-\infty$ and decrease like $(-2/\sigma)^{-k}$. The associated last term $S_N(C_{\bar{1}})$ is of the negligible order $e^{-N \log N}$. For the cuts $C_{2,\bar{2},3}$, the integral (17) starts at $\hat{g} = 1/\sigma$ and has therefore the leading behavior

$$\varepsilon_k(C_{2,\bar{2},3}) \sim \sigma^k, \tag{37}$$

yielding at first a contribution to the Nth term in the reexpansion of the order of

$$S_N(C_{2,\bar{2},3}) \sim (\sigma \hat{g})^N, \tag{38}$$

which decreases merely like (34) and does not explain the empirically observed convergence in the strong-coupling limit. The important additional information discovered in [7,8] is that the cuts in Fig. 1 do not really reach the point $\sigma\hat{g} = 1$. There exists a small circle of radius $\Delta\hat{g} > 0$ in which $\hat{E}(\hat{g})$ has no singularities at all. This is a consequence of the fact unused up to this point that the strong-coupling expansion (8) converges for $g_0 > g_\mathrm{s}$. For the reduced function $\hat{E}(\hat{g}_0)$, this expansion reads:

$$\hat{E}(\hat{g}) = (\hat{g}_0)^{p/q} \left\{ b_0 + b_1 \left[\frac{\hat{g}_0}{(1-\sigma\hat{g}_0)^{q/2}} \right]^{-2/q} + b_2 \left[\frac{\hat{g}_0}{(1-\sigma\hat{g}_0)^{q/2}} \right]^{-4/q} + \ldots \right\}. \tag{39}$$

The convergence of (8) for $g_0 > g_\mathrm{s}$ implies that (39) converges for all $\sigma\hat{g}_0$ in a neighborhood of the point $\sigma\hat{g}_0 = 1$ with a radius

$$\Delta(\sigma\hat{g}_0) = \left| \frac{\hat{g}_0}{\bar{g}_\mathrm{s}} \right|^{2/q}, \tag{40}$$

where $\bar{g}_\mathrm{s} \equiv g_\mathrm{s}/\kappa^q$. For large N, the denominator K^q in $\hat{g}_0$ on the right-hand side makes $\Delta(\sigma\hat{g}_0)$ go to zero like

$$\Delta(\sigma\hat{g}_0) \approx \frac{1}{(N|\bar{g}_s|c)^{2/q}}. \tag{41}$$

Thus the integration contours of the moment integrals (17) for the contributions $\varepsilon_k(C_i)$ of the other cuts do not begin at the point $\sigma\hat{g} = 1$, but a little distance $\Delta(\sigma\hat{g})$ away from it. If $q < 4$, i.e. if $\omega > 1/2$, the intersection points of the small circle with the cuts C_2 and $C_{\bar{2}}$ have a real part larger than unity. This produces a suppression factor to the previous result (37) of the integral (17)

$$(\sigma\hat{g}_0)^{-N} \sim [1 + \Delta(\sigma\hat{g}_0)]^{-N}. \tag{42}$$

bringing the last term of the series S_N to

$$S_N(C_{2,\bar{2},3}) \sim (\sigma\hat{g}_0)^N \frac{1}{[1+\Delta(\sigma\hat{g}_0)]^N}. \tag{43}$$

instead of (38). Inserting (41), we find that this goes to zero with the same characteristic behavior (36) as the contribution from the cut C_1:

$$S_N(C_{2,\bar{2},3}) \approx e^{-c'''N^{1-\omega}}, \qquad c''' > 0. \tag{44}$$

Such a behavior characterizes therefore the convergence for $N \to \infty$, which will be needed in Section VI to extrapolate finite-N results to $N \to \infty$.

IV. ADAPTATION TO THE PROBLEM AT HAND

In the examples treated in the original paper [9], the strong-coupling parameters p and q were known. In the problem here, this is not be the case. They can, however, be easily determined. We simply observe that p/q and $2/q$ can be found from the strong-coupling limits of an infinite set of logarithmic derivatives of $W_N(g_0)$:

$$\frac{p}{q} = F_1(\infty), \qquad F_1(g_0) \equiv \frac{d\log W_N(g_0)}{d\log g_0} = g_0 \frac{W'_N(g_0)}{W_N(g_0)}, \tag{45}$$

$$-\frac{2}{q} - 1 = F_2(\infty), \qquad F_2(g_0) \equiv \frac{d\log F'_1(g_0)}{d\log g_0} = g_0 \frac{F''_1(g_0)}{F'_1(g_0)}, \tag{46}$$

$$-\frac{2}{q} - 1 = F_3(\infty), \qquad F_3(g_0) \equiv \frac{d\log F'_2(g_0)}{d\log g_0} = g_0 \frac{F''_2(g_0)}{F'_2(g_0)}, \tag{47}$$

$$\vdots \; .$$

If the parameter p happens to be zero, there is a further sequence of formulas for the parameter q:

$$-\frac{2}{q}-1=G_1(\infty),\qquad G_1(g_0)\equiv\frac{d\log W_N'(g_0)}{d\log g_0}=g_0\frac{W_N''(g_0)}{W_N'(g_0)},\tag{48}$$

$$-\frac{2}{q}-1=G_2(\infty),\qquad G_2(g_0)\equiv\frac{d\log G_1'(g_0)}{d\log g_0}=g_0\frac{G_1''(g_0)}{G_1'(g_0)},\tag{49}$$

$$-\frac{2}{q}-1=G_3(\infty),\qquad G_3(g_0)\equiv\frac{d\log G_2'(g_0)}{d\log g_0}=g_0\frac{G_2''(g_0)}{G_2'(g_0)},\tag{50}$$

$$\vdots\quad .$$

Formulas (45) and (48) will be crucial to the development in Section VI.

V. PERTURBATION EXPANSIONS

We shall apply our technique to the renormalization constants of the ϕ^4-theory with the bare euclidean action

$$\mathcal{A}=\int d^D\left\{\frac{1}{2}\left[\partial\phi_0(x)\right]^2+\frac{1}{2}m_0^2\phi_0^2(x)+\frac{48\pi}{n+8}\frac{g_0}{4!}\left[\phi_0^2(x)\right]^2\right\}\tag{51}$$

in $D=3$ dimension. The field $\phi_0(x)$ is an n-dimensional vector, and the action is O(n)-symmetric in this vector space. The Ising model corresponds to $n=1$, the critical behavior of percolation is described by $n=0$, superfluid phase transitions by $n=2$, and classical Heisenberg magnetic systems by $n=3$.

By calculating the Feynman diagrams up to six loops, one obtains renormalized values of mass, coupling constant, and field related to the bare input quantities by renormalization constants Z_ϕ, Z_m, Z_g:

$$m_0^2=m^2\ Z_mZ_\phi^{-1},\qquad \lambda_0=\lambda\,Z_gZ_\phi^{-2},\qquad \phi_0=\phi\ Z_\phi^{1/2}.\tag{52}$$

The divergences are removed by analytic regularization [15]. In the literature, one finds expansions for certain logarithmic derivatives of the renormalization factors (the so-called *renormalization group functions*) up to six loops. Introducing the reduced dimensionless coupling constants $\bar g\equiv g/m$ and $\bar g_0\equiv g_0/m$, these can be written as

$$\omega(\bar g)=\frac{d\beta(\bar g)}{d\bar g},\qquad \beta(\bar g)=-\left\{\frac{d}{d\bar g}\log[\bar gZ_g/Z_\phi^2]\right\}^{-1}=-\bar g_0\left(\frac{d\bar g_0}{d\bar g}\right)^{-1},\tag{53}$$

$$\eta(\bar g)=\beta(\bar g)\frac{d}{d\bar g}\log Z_\phi=\beta(\bar g)\frac{d}{d\bar g}\log\frac{\phi_0^2}{\phi^2},\tag{54}$$

$$\eta_m(\bar g)=-\beta(\bar g)\frac{d}{d\bar g}\log[Z_m^2/Z_\phi]=-\beta(\bar g)\frac{d}{d\bar g}\log\frac{m_0^2}{m^2}.\tag{55}$$

The function $\beta(\bar g)$ may be considered as a function of the reduced bare coupling constant $\bar g_0$. As such it is equal to the logarithmic derivative of the function $\bar g=\bar g(\bar g_0)$:

$$\beta(\bar g_0)=-\frac{d\ \bar g(\bar g_0)}{d\log\bar g_0}=-\bar g_0\hat g'(\bar g_0).\tag{56}$$

The function $\omega(\bar g)$ may then be obtained from the function $\bar g=\bar g(\bar g_0)$ by the logarithmic derivative

$$\omega(\bar g_0)\ =\ \frac{d\log\beta(\bar g_0)}{d\log\bar g_0}=-\frac{d\log[\bar g_0\bar g'(\bar g_0)]}{d\log\bar g_0}\ =\ -1-g_0\frac{\bar g''(\bar g_0)}{\bar g'(\bar g_0)}.\tag{57}$$

Comparison with Eq. (48) shows that if $\bar g(\bar g_0)$ goes to a constant $\bar g^*$ in the strong-coupling limit $\bar g_0\to\infty$, the limiting value $\omega\equiv\omega(\infty)$ plays the role of a parameter of approach $2/q$ of the strong-coupling expansion of the function $\bar g(\bar g_0)$.

Similarly we convert Eqs. (54) and (55) into functions of the bare coupling constant g_0:

$$\eta(\bar g_0)=\frac{d}{d\log\bar g_0}\log\frac{\phi^2}{\phi_0^2},\qquad \eta_m(\bar g_0)=-\frac{d}{d\log\bar g_0}\log\frac{m^2}{m_0^2}.\tag{58}$$

If $\eta(\bar g_0)$ and $\eta_m(\bar g_0)$ have finite strong-coupling limits $\eta=\eta(\infty)$ and $\eta_m=\eta_m(\infty)$, these equations imply the strong-coupling behavior

$$\frac{\phi^2}{\phi_0^2} \approx \bar{g}_0^{\eta}, \qquad \frac{m^2}{m_0^2} \approx \bar{g}_0^{-\eta_m}. \tag{59}$$

By replacing the reduced coupling constant $\bar{g}_0$ by g_0/m, this implies the small-mass behavior at a fixed bare coupling g_0:

$$\frac{\phi^2}{\phi_0^2} = \text{const} \times m^{-\eta}, \qquad \frac{m^2}{m_0^2} = \text{const} \times m^{\eta_m}. \tag{60}$$

In the field-theoretic description of second-order phase transitions, the bare square mass m_0^2 vanishes near the critical temperature like $\tau \propto (T - T_c)$. For the renormalized mass m, we obtain from the second equation in (60) the scaling relation $m \propto \tau^{1/(2-\eta_m)}$. Experiments observe that the coherence length of fluctuations $\xi = 1/m$ increases near T_c like $\tau^{-\nu}$, so that we derive for the critical exponent ν a value $1/(2-\eta_m)$. Similarly we see from the first equation in (60) that the scaling dimension $D/2 - 1$ of the bare field ϕ_0 for $T \to T_c$ is changed, in the strong-coupling limit $g_0 \to \infty$, to $D/2 - 1 + \eta/2$, the number η being the so-called anomalous dimension of the field. This implies a change in the large-distance behavior of the correlation functions $\langle \phi(x)\phi(0) \rangle$ at T_c from the free-field behavior r^{-D+2} to $r^{-D+2-\eta}$. The magnetic susceptibility is determined by the integrated correlation function $\langle \phi_0(x)\phi_0(0) \rangle$. At zero coupling constant g_0, this is proportional to $1/m_0^2 \propto \tau^{-1}$, which is changed by fluctuations to $m^{-2}\phi_0^2/\phi^2$. This has a temperature behavior $m^{-(2-\eta)} = \tau^{-\nu(2-\eta)} \equiv \tau^{-\gamma}$, which defines the critical exponent $\gamma = \nu(2-\eta)$ observable in magnetic experiments.

The following expansions are available in the literature [10–12]:

$$\begin{aligned}
\omega(\bar{g}) &= -1 + 2\bar{g}\,(8+n) - 3\bar{g}^2\,(760/27 + 164n/27) + 4\bar{g}^3\left(199.640417 + 54.94037698n + 1.34894276n^2\right) \\
&\quad +5\bar{g}^4\left(-1832.206732 - 602.5212305n - 35.82020378n^2 + 0.15564589n^3\right) \\
&\quad +6\bar{g}^5\left(20770.17697 + 7819.564764n + 668.5543368n^2 + 3.2378762n^3 + 0.05123618n^4\right) \\
&\quad +7\bar{g}^6\left(-271300.0372 - 114181.4357n - 12669.22119n^2 - 265.8357032n^3 + 1.07179839n^4 + 0.02342417n^5\right) \\
\eta(\bar{g}) &= \bar{g}^2\,(16/27 + 8n/27) + \bar{g}^3\left(0.3949440224 + 0.246840014n + 0.0246840014n^2\right) \\
&\quad +\bar{g}^4\left(6.512109933 + 4.609221057n + 0.6679859202n^2 - 0.0042985626n^3\right) \\
&\quad +\bar{g}^5\left(-21.64720643 - 15.1880934n - 1.891139282n^2 + 0.1324510614n^3 - 0.0065509222n^4\right) \\
&\quad +\bar{g}^6\left(369.7130739 + 300.7208933n + 64.07744656n^2 + 3.054030987n^3 - 0.0203994485n^4 - 0.0055489202n^5\right) \\
\eta_m(\bar{g}) &= \bar{g}\,(2+n) - \bar{g}^2\,(92/27 + 46n/27) + \bar{g}^3\left(18.707787762 + 12.625201157n + 1.6356536385n^2\right) \\
&\quad +\bar{g}^4\left(-134.28726152 - 98.33833174n - 15.117303198n^2 + 0.2400236453n^3\right) \\
&\quad +\bar{g}^5\left(1318.4281763 + 1046.8184247n + 209.71327323n^2 + 8.143135609n^3 + 0.0937915707n^4\right) \\
&\quad +\bar{g}^6\left(-15281.544489 - 12918.644832n - 2980.2279474n^2 - 164.6575873n^3 + 3.0931477063n^4 + 0.0495801299n^5\right).
\end{aligned} \tag{61}$$

Here and in the subsequent set of perturbative expansions, we save space by omitting in each term $\bar{g}^n$ a denominator $(n+8)^n$.

By integrating (53), we see that $\omega(\bar{g})$ implies the relation between bare and renormalized coupling constant:

$$\begin{aligned}
g_0 &= g\big[1 + \bar{g}\ (8+n) + \bar{g}^2\ \left(1348/27 + 350n/27 + n^2\right) \\
&\quad +\bar{g}^3\ \left(315.8307562667 + 120.7825947383n + 17.3632278267n^2 + n^3\right) \\
&\quad +\bar{g}^4\ \left(1813.1642655362 + 949.9400421368n + 203.4347168377n^2 + 21.5210551192n^3 + n^4\right) \\
&\quad +\bar{g}^5\ \left(11664.58684418 + 7259.6266136476n + 1965.0940131759n^2 + 298.9857773851n^3 + 25.5436671032n^4 + n^5\right) \\
&\quad +\bar{g}^6\ \left(57253.8939657167 + 47753.8060061961n + 16981.2530394653n^2 + 3357.7450242179n^3\right. \\
&\qquad \left. + 407.679442164n^4 + 29.4800395765n^5 + n^6\right)\big].
\end{aligned} \tag{62}$$

This can be inverted to

$$\begin{aligned}
\bar{g} &= \bar{g}_0\big[1 - \bar{g}_0\ (8+n) + \bar{g}_0^2\ \left(2108/27 + 514n/27 + n^2\right) \\
&\quad +\bar{g}_0^3\ \left(-878.7937193 - 312.63444671n - 32.54841303n^2 - n^3\right) \\
&\quad +\bar{g}_0^4\ \left(11068.06183 + 5100.403285n + 786.3665699n^2 + 48.21386744n^3 + n^4\right) \\
&\quad +\bar{g}_0^5\ \left(-153102.85023 - 85611.91996n - 17317.702545n^2 - 1585.1141894n^3 - 65.82036203n^4 - n^5\right)
\end{aligned}$$

$$
\begin{aligned}
&+\bar{g}_0^6\ \big(2297647.148\ + 1495703.313\,n + 371103.0896n^2 + 44914.04818n^3\\
&\qquad + 2797.291579n^4 + 85.21310501n^5 + n^6\big)\ ,].
\end{aligned}
\tag{63}
$$

where we suppress a denominator $(n+8)^n$ in each term $\bar{g}_0^n$. Inserting this into the functions (61), they become

$$
\begin{aligned}
\omega(\bar{g}_0) = &-1 + 2\bar{g}_0\ (8+n) - \bar{g}_0^2\ \big(1912/9 + 452n/9 + 2n^2\big)\\
&+\bar{g}_0^3\ \big(3398.857964 + 1140.946693n + 95.9142896n^2 + 2n^3\big)\\
&+\bar{g}_0^4\ \big(-60977.50127 - 26020.14956n - 3352.610678n^2 - 151.1725764n^3 - 2n^4\big)\\
&+\bar{g}_0^5\ \big(1189133.101 + 607809.998n + 104619.0281n^2 + 7450.143951n^3 + 214.8857494n^4 + 2n^5\big)\\
&+\bar{g}_0^6\ \big(-24790569.76 - 14625241.87n - 3119527.967n^2\\
&\qquad - 304229.0255n^3 - 14062.53135n^4 - 286.3003674n^5 - 2n^6\big)\ ,
\end{aligned}
\tag{64}
$$

$$
\begin{aligned}
\eta(\bar{g}) = &\ \bar{g}_0^2\ (16/27 + 8n/27) + \bar{g}_0^3\ \big(-9.086537459 - 5.679085912n - 0.5679085912n^2\big)\\
&+\bar{g}_0^4\ \big(127.4916153 + 94.77320534n + 17.1347755n^2 + 0.8105383221n^3\big)\\
&+\bar{g}_0^5\ \big(-1843.49199 - 1576.46676n - 395.2678358n^2 - 36.00660242n^3 - 1.026437849n^4\big)\\
&+\bar{g}_0^6\ \big(28108.60398 + 26995.87962n + 8461.481806n^2 + 1116.246863n^3 + 62.8879068n^4 + 1.218861532n^5\big)
\end{aligned}
\tag{65}
$$

$$
\begin{aligned}
\eta_m(\bar{g}) = &\ \bar{g}_0\ (2+n) - \bar{g}_0^2\ \big(-523/27 - 316n/27 - n^2\big)\\
&+\bar{g}_0^3\ \big(229.3744544 + 162.8474234n + 26.08009809n^2 + n^3\big)\\
&+\bar{g}_0^4\ \big(-3090.996037 - 2520.848751n - 572.3282893n^2 - 44.32646141n^3 - n^4\big)\\
&+\bar{g}_0^5\ \big(45970.71839 + 42170.32707n + 12152.70675n^2 + 1408.064008n^3 + 65.97630108n^4 + n^5\big)\\
&+\bar{g}_0^6\ \big(-740843.1985 - 751333.064n - 258945.0037n^2 - 39575.57037n^3 - 2842.8966n^4 - 90.7145582n^5 - n^6\big)\ .
\end{aligned}
$$

Let us also write down a power series expansion for the function $\gamma(g_0) = [2-\eta(g_0))/[2-\eta_m(g_0)]$ which tends to the critical exponent γ of susceptibility. In resummation procedures applied to functions of the renormalized coupling constant g, this series has always been favored over that for $\eta(g)$ since, in contrast to $\eta(g)$, its expansion coefficients of $\gamma(g)$ have alternating signs permitting application of Padé-Borel resummation techniques [16]. For $\gamma(g_0)$, the series reads:

$$
\begin{aligned}
\gamma(g) = &\ 1 + \bar{g}_0\ (2+n)/2 + \bar{g}_0^2\ \big(-9 - 5n - n^2/4\big)\\
&+\bar{g}_0^3\ \big(100.5267922 + 64.05955095n + 7.148077413n^2 + 0.125n^3\big)\\
&+\bar{g}_0^4\ \big(-1306.696473 - 953.5355208n - 165.6165894n^2 - 7.886473674n^3 - 0.0625n^4\big)\\
&+\bar{g}_0^5\ \big(19047.24345 + 15717.20743n + 3667.58258n^2 + 300.9668324n^3 + 7.848484825n^4 + 0.03125n^5\big)\\
&+\bar{g}_0^6\ \big(-304324.882 - 279842.9929n - 81107.12259n^2 - 9519.124419n^3 - 457.7147389n^4\\
&\qquad - 7.463312096n^5 - 0.015625n^6\big)
\end{aligned}
\tag{66}
$$

VI. FROM WEAK TO STRONG COUPLINGS

We are now ready to apply our theory of Sections II–IV to these expansions. First we study the $\bar{g}_0 \to \infty$ -limit of the series (63) for the renormalized reduced coupling constant $\bar{g}$. We expect the theory to reproduce experimentally observed scaling laws which means that $\bar{g}(\bar{g}_0)$ should tend to some constant value: $\bar{g}(\bar{g}_0) \to \bar{g}^*$ for $\bar{g}_0 \to \infty$. In the notation of (8), the parameter p/q for the leading power behavior in $\bar{g}_0$ must therefore be equal to zero.

The approach parameter $2/q$ is unknown. It can, however, be determined from Eq. (48). Inserting for $W_N(\bar{g}_0)$ the function $g(\bar{g}_0)$, we calculate the critical exponent $\omega = 2/q$ for O(n)-symmetric theories with $n = 0,\ 1,\ 2,\ 3,\ \ldots$. The highest available approximation ω_6 is found to yield the values given in the last column of Table II.

Since N is not very large, the results require extrapolation to infinite N. The functional form of the N-dependence was determined in (44), predicting the large-N behavior

$$
\omega_N \approx \omega - b\, e^{-cN^{1-\omega}}\, .
\tag{67}
$$

In Fig. 2 we illustrate how the successive approximations ω_2–ω_6 are fitted by this asymptotic curve for O(n)-theories with $n = 0$ (percolation), $n = 1$, (Ising), $n = 2$ (superfluid Helium), $n = 3$ (classical Heisenberg model). The extrapolated $N \to \infty$ -values obtained by such fits are shown in the second-last column of Table II. They are plotted in the first of Figs. 3, together with the sixth-order approximation to show the significance of the $N \to \infty$ -extrapolation. Our numbers merge smoothly with the $1/n$-expansion curve which has been calculated only recently to order $1/n^2$ [17]:

$$\omega = 1 - 8\frac{8}{3\pi^2}\frac{1}{n} + 2\left(\frac{104}{3} - \frac{9}{2}\pi^2\right)\left(\frac{8}{3\pi^2}\frac{1}{n}\right)^2 + \mathcal{O}(n^{-3}). \tag{68}$$

To judge the internal consistency of our procedure, we calculate ω once more from the series (48) for $\omega = 2/q$. Since ω appears also on the right-hand side of the series via the parameter q, this represents a self-consistency relation which can be iterated until input and output values for ω agree. The results are shown in Fig. 4 for increasing orders $N = 2,\ 3,\ 4,\ 5,\ 6$, in the Ising case $n = 1$. The data points are again fitted with the functional behavior (67), using the same extrapolated $\omega(\infty)$-values as in the previous fits.

The critical exponents ω can also be calculated from the expansions for γ and ν in this self-consistent way, as shown in Fig. 5.

Note that the agreement between the self-consist ω-values with the previous ones determined from the $p = 0$ -condition can be considered as a confirmation of the hypothesis that the thery does indeed have a definite strong-coupling limit in which $g(g_0)$ tends to a constant g^* (an *infrared-stable fixed point* in the language of the renormalization group). It also implies all other scaling properties to be derived in the sequel.

Proceeding to other critical exponents, we now take the function $\nu^{-1}(\bar{g}_0) = 2 - \eta_m(\bar{g}_0)$ to the strong-coupling limit, to determine $\nu = \nu(\infty)$. The extrapolations to large N are done with the help of the approximations $\nu_2,\ \nu_3, \dots, \nu_6$, as illustrated in Figs. 6. The resulting critical exponents are plotted against n in the second of Figs. 3, and listed in Table II. The points are seen to merge well with those of the $1/n$ -expansion for ν [18]:

$$\nu = 1 - 4\frac{8}{3\pi^2}\frac{1}{n} + \left(\frac{56}{3} - \frac{9}{2}\pi^2\right)\left(\frac{8}{3\pi^2}\frac{1}{n}\right)^2 + \mathcal{O}(n^{-3}).$$

The function $\gamma(\bar{g}_0)$ is treated somewhat differently. Since γ serves to determine the critical exponent η via the scaling relation $\eta = 2 - \gamma/\nu$, and since this combination is very sensitive to small errors in γ (and in ν), we proceed as follows: After applying our method to the γ-series and calculating the approximations $\gamma_2,\ \gamma_3, \dots, \gamma_6$, we go over to $\eta_2,\ \eta_3, \dots, \eta_6$ via the scaling relation $\eta_N = 2 - \gamma_N/\nu$, and perform the extrapolation $N \to \infty$ on these η-values, as illustrated in Figs. 7. In this way we obtain the smooth η-curves shown in the fourth of Figs. 3 and listed in Table II. The values of the highest approximation η_6 in the parentheses of Table II are obtained from a direct variational treatment of the perturbation expansion for η. These are closer to the final η-values than the approximations of $\eta_6 = 2 - \gamma_6/\nu$ used for the exprapolation, which are indicated in the parenthesis on top of Figs. 7. Still, we have used the latter for extrapolation since there are five of them to be fitted with the asymtotic curve (67), while the η-expansion which has no linear term in $\hat{g}_0$ provides us only with four values, making a fit less reliable.

The extrapolated η-values are inserted into the scaling relation $\gamma = \nu(2 - \eta)$ to derive the extrapolated γ-values plotted in the third of Figs. 3 and listed in Table II. A direct extrapolation $N \to \infty$ of the variational approximations γ_N to the γ-expansion turn out to be fully compatible with the previous results, as illustrated in Fig. 8.

For large n, our critical exponents η are in excellent agreement with the $1/n$ -expansion, which is known up to order $1/n^3$ [19]:

$$\eta = \frac{8}{3\pi^2}\frac{1}{n} - \frac{28}{3}\left(\frac{8}{3\pi^2}\frac{1}{n}\right)^2 - \left[\frac{653}{18} - \left(27\log 2 + \frac{47}{4}\right)\zeta(2) + \frac{189}{4}\zeta(3)\right]\left(\frac{8}{3\pi^2}\frac{1}{n}\right)^3 + \mathcal{O}(n^{-4}), \tag{69}$$

where $\zeta(x)$ is Riemann's zeta function. Note that for η, the finite-N corrections are very small. In fact, from the η_6-values near $n = 1000$ we can extract numerically a $1/n$-expansion

$$\eta \approx 0.303\,\frac{1}{n} - 0.104\,\frac{1}{n^2}, \tag{70}$$

which agrees reasonably well with the exact expansion

$$\eta \approx 0.270\,\frac{1}{n} - 0.195\,\frac{1}{n^2}. \tag{71}$$

It is worth pointing out that we may apply our strong-coupling theory also to $1/n$ -expansions for η, to find expressions valid for all n down to $n = 0$ by treating $1/n$ just like the variable g_0 in Section II. Taking the $1/n$ -expansion as an example, we derive a smooth fit from large to rather small n by adding another term $-104/n^4$ to

(69) before going through the resummation. The extra term improves the fit. Since η starts out linearly at the "strong-coupling" value $1/n = \infty$, the prameters p and q are 0 and 2. The resulting curve is shown in the fourth of Figs. 3. It fits all data, except for very small n. The figure shows also successive approximations provided by the $1/n$-expansion.

Also γ merges well with its $1/n$-expansion

$$\gamma = 2 - 9\,\frac{8}{3\pi^2}\frac{1}{n} + \left(44 - 9\pi^2\right)\left(\frac{8}{3\pi^2}\frac{1}{n}\right)^2 + \mathcal{O}(n^{-3}), \tag{72}$$

as seen in the third of Figs. 3.

Finally, we exhibit the full power of our theory by plotting for the example of the Ising case $n = 1$ the functions $\omega(\bar{g}_0)$, $\nu(\bar{g}_0)$, $\eta(\bar{g}_0)$ for all coupling strengths in Figs 9–12, together with the diverging perturbative approximations as well as the convergent strong-coupling expansion. We do this successively for each increasing order of the approximations. On a logarithmic plot, the quality of these very different approximations looks surprisingly similar.

Although the functions $\bar{g}(\bar{g}_0)$, $\omega(\bar{g}_0)$, $\nu(\bar{g}_0)$, $\eta(\bar{g}_0)$ in Figs. 9–12 were derived by a numerical variational procedure, it is possible to write the results down in the form of new strong-coupling expansions which converge for *all* coupling strengths. We shall demonstrate this only for the Ising case $n = 1$, where $\omega = 0.805$. Consider first the function $\bar{g}(\bar{g}_0)$ with the sixth-order strong-coupling expansion (8) (whose $\bar{g}_0$-dependence is displayed in Fig. 9):

$$\begin{aligned}\bar{g} = {}& 1.400036164909792 - 2.015076019427151/\bar{g}_0^{\omega} + 2.512390732560552/\bar{g}_0^{2\omega} - 2.903034628806387/\bar{g}_0^{3\omega} \\ &+3.123423917471507/\bar{g}_0^{4\omega} - 3.108796470872297/\bar{g}_0^{5\omega} + 2.844130229268904/\bar{g}_0^{6\omega} - 2.38207097645026/\bar{g}_0^{7\omega}.\end{aligned} \tag{73}$$

For $\bar{g}_0 \to \infty$, this goes against the fixed point $\bar{g}^* = 1.400036164909792$, the critical coupling constant. The important observation is now that by changing variables to $x = x(\bar{g}_0)$ defined by

$$\bar{g}_0 \equiv (1-x)/x^{1/\omega}, \tag{74}$$

and reexpanding up to the order x^6, we obtain a new modified strong-coupling expansion

$$\begin{aligned}\bar{g}(x) = {}& 1.400036164909792 - 2.015076019427151x + 0.890254536921696x^2 - 0.322063465947966x^3 \\ &+0.02243448556302718x^4 + 0.02753503840016558x^5 + 0.004440133592881424x^6 - 0.005052466607713413x^7.\end{aligned} \tag{75}$$

Numerically, this converges for *all* $\bar{g}_0 \in (0,\infty)$ where $x \in (1,0)$. Indeed, when plotting this function in Fig. 9, it falls right on top of the previously calculated curve representing the full variational expression (4). To verify the convergence for small couplings, we insert $x = 1$ and obtain 0.0025, which misses only slightly the free-field value 0.

Similarly we obtain a strong-coupling expansion of the critical exponent $\nu(\bar{g}_0)$. In accordance with the above determination of ν from a resummation of the series for the inverse $\nu^{-1}(\bar{g}_0) = 2 - \eta_m(\bar{g}_0)$ we first derive the strong-coupling expansion for the inverse, and invert the resulting power series. The result is

$$\begin{aligned}\nu(\bar{g}_0) = {}& 0.6264612502473953 - 0.2094930499887895/\bar{g}_0^{\omega} + 0.3174116223980956/\bar{g}_0^{2\omega} - 0.4674929704457459/\bar{g}_0^{3\omega} \\ &+0.6755577434122434/\bar{g}_0^{4\omega} - 0.961768075384486/\bar{g}_0^{5\omega} + 1.35507538545696/\bar{g}_0^{6\omega}.\end{aligned} \tag{76}$$

Changing again to the variable $x = x(\bar{g}_0)$, and reexpanding up to the order x^6, we obtain the convergent series

$$\begin{aligned}\nu(x) = {}& 0.6264612502473953 - 0.2094930499887895x + 0.1487697171571201x^2 - 0.1086595778647923x^3 \\ &+0.07115354531150436x^4 - 0.04710070728711805x^5 + 0.0316053426101743x^6.\end{aligned} \tag{77}$$

A plot in Fig. 11 is once more indistinguishable from the previous curve for all $\bar{g}_0$. At $x = 1$, this series gives now 0.5127 which is only two percent larger than the free-field value 1/2.

By inverting the series (75), we find x as a function of the deviation $\Delta\bar{g} \equiv \bar{g} - \bar{g}^*$ of the renormalized coupling from its strong-coupling value $\bar{g}^*$:

$$\begin{aligned}x(\bar{g}) = {}& -0.4962591933798516\Delta\bar{g} + 0.1088027548080915\Delta\bar{g}^2 - 0.02817579418977608\Delta\bar{g} + 0.005412315522686147\Delta\bar{g}^4 \\ &+0.00005845448187202598\Delta\bar{g}^5 - 0.0006265703321966366\Delta\bar{g}^6 + 0.0002698406382563225\Delta\bar{g}^7.\end{aligned} \tag{78}$$

After inserting this into (77), we obtain the critical exponent ν as a power series in:

$$\begin{aligned}\nu(\bar{g}) = {}& 0.6264612502473953 + 0.1039628520061216\Delta\bar{g} + 0.01384457142352891\Delta\bar{g}^2 + 0.003117046054051534\Delta\bar{g}^3 \\ &+0.0003684746290205897\Delta\bar{g}^4 + 8.64736629778199\times10^{-5}\Delta\bar{g}^5 - 7.632599338747235\times10^{-6}\Delta\bar{g}^6.\end{aligned} \tag{79}$$

In the weak-coupling limit, this is equal to 0.50039 rather than the exact 1/2.

We now turn to $\gamma(\bar{g}_0)$, for which the initial strong-coupling expansion (8) reads

$$\gamma(\bar{g}_0) = 1.400036164909792 - 2.015076019427151/\bar{g}_0^{\omega} + 2.512390732560552/\bar{g}_0^{2\omega} - 2.903034628806387/\bar{g}_0^{3\omega} \\ +3.123423917471507/\bar{g}_0^{4\omega} - 3.108796470872297/\bar{g}_0^{5\omega} + 2.844130229268904/\bar{g}_0^{6\omega} - 2.38207097645026/\bar{g}_0^{6\omega}, \quad (80)$$

which goes over into the following convergent new strong-coupling expansions in the variables x and $\Delta\bar{g}$:

$$\gamma(x) = 1.234453309456454 - 0.3495068067938822x + 0.1754449364006056x^2 \\ -0.07496457184890283x^3 \\ +0.01553288625746019x^4 - 0.0007559094604789874x^5 - 0.00005490359805215839x^6 \quad (81)$$

$$\gamma(\bar{g}) = 1.234453309456454 + 0.1734459660202996\Delta\bar{g} + 0.005180080229495593\Delta\bar{g}^2 + 0.0000633752908487349\Delta\bar{g}^3 \\ +7.604216767658758 * 10^-6\Delta\bar{g}^4 + 0.00003970285111914907\Delta\bar{g}^5 - 0.00006597501217820878\Delta\bar{g}^6 \quad (82)$$

Combining these with (76), (77), and (79), we derive from the scaling relation $\eta = 2 - \gamma/\nu$ the corresponding expansions for η:

$$\eta(\bar{g}_0) = 0.02948177725444778 - 0.101048653404458/\bar{g}_0^{\omega} + 0.2354470919307273/\bar{g}_0^{2\omega} - 0.452249390223013/\bar{g}_0^{3\omega} \\ +0.7620101849621524/\bar{g}_0^{4\omega} - 1.160617256113553/\bar{g}_0^{5\omega} + 1.627651002530155/\bar{g}_0^{6\omega}, \quad (83)$$

$$\eta(x) = 0.02948177725444778 - 0.101048653404458x + 0.1541029259401388x^2 - 0.1465926820210476x^3 \\ +0.0958727624055483x^4 - 0.04186840682453081x^5 + 0.01220597704767396x^6, \quad (84)$$

$$\eta(\bar{g}) = 0.02948177725444778 + 0.05014632323061653\Delta\bar{g} + 0.02695704683941055\Delta\bar{g}^2 + 0.004121619548050133\Delta\bar{g}^3 \\ -0.00038235521822759\Delta\bar{g}^4 + 0.00001736700551847262\Delta\bar{g}^5 + 0.00004358809862557 14\Delta\bar{g}^6. \quad (85)$$

The weak-coupling limits of these expansions are $\gamma(x=0) = 1.00015$, $\gamma(\bar{g}=0) = 1.00092$ rather than the exact 1, and $\eta(x=0) = 0.0025$, $\eta(\bar{g}=0) = 0.00043$ rather than the exact 0. The expansions for $\eta(x)$ converge rather slowly, so it is preferable to do calculations involving η by replacing it by $\eta = 2 - \gamma/\nu$ and using the expansions for γ and ν without reexpanding the ratio γ/ν. Then the plots of the convergent expansion for $\gamma(x)$ and of $\eta(x)$ are found once more to be very close to the plots of the corresponding full sixth-order approximations in Figs. 11 and 12.

It is now easy to give convergent expansions for the full m_0 dependence of the renormalization factors. From (58) we see that

$$\frac{\phi^2}{\phi_0^2} = \exp\left[\int_0^{\bar{g}_0} \frac{d\bar{g}_0'}{\bar{g}_0'} \eta(\bar{g}_0')\right] = \exp\left[\int_x^1 \frac{dx'}{x'} \frac{\eta(x')}{f(x')}\right], \quad (86)$$

$$\frac{m^2}{m_0^2} = \exp\left[-\int_0^{\bar{g}_0} \frac{d\bar{g}_0'}{\bar{g}_0'} \eta_m(\bar{g}_0')\right] = \frac{m^2}{g_0^2} \exp\left[-\int_x^1 \frac{dx'}{x'} \frac{\eta_m(x')}{f(x')}\right]. \quad (87)$$

where we have introduced the function

$$f(x) \equiv -\frac{d\log x}{d\log \bar{g}_0} = \omega \frac{1-x}{1-(1-\omega)x}. \quad (88)$$

After a subtraction of the logarithmic divergence of the integrals at $x=0$, these can be rewritten as

$$\frac{\phi^2}{\phi_0^2} = x^{\eta/\omega} e^{-I(x)}, \qquad \frac{m^2}{m_0^2} = x^{\eta_m/\omega} e^{-I_m(x)}, \quad (89)$$

where $I(x)$ and $I_m(x)$ are the finite integrals

$$I(x) = \int_x^1 \frac{dx'}{x'} \left[\frac{\eta(x')}{f(x')} - \frac{\eta}{\omega}\right], \qquad I_m(x) = \int_x^1 \frac{dx'}{x'} \left[\frac{\eta_m(x')}{f(x')} - \frac{\eta_m}{\omega}\right]. \quad (90)$$

The integral $I_m(x)$ can readily be performed using a power series for $\eta_m(x)$ obtained from (77) via $\eta_m(x) = 2 - \nu^{-1}(x)$. The result is

$$I_m(x) = 0.3023858220717581 - 0.3374211153180052x + 0.05257791758557147x^2 - 0.02006073290035280x^3 \\ +0.002720917209250741x^4 - 0.0001454953425150762x^5 - 0.00005731330570724153x^6. \quad (91)$$

By combining the second equation in (89) with (74) written as

$$\frac{m^2}{g_0^2} = x^{2/\omega}(1-x)^{-2}, \tag{92}$$

we find

$$\frac{m_0^2}{g_0^2} = x^{1/\nu\omega}(1-x)^{-2}e^{I_m(x)}. \tag{93}$$

In the weak-coupling limit $x \to 1$, the integral vanishes and the renormalized mass approaches the bare mass. In the strong-coupling limit $x \to 0$, on the other hand, $I_m(x)$ becomes a constant and we obtain once more the scaling relation $m \propto (m_0^2)^\nu$. To study the crossover from the weak to the strong-coupling behavior in a way which makes contact with critical phenomena, we introduce the *Ginzburg temperature interval* ΔT_G within which fluctuations are important. and set

$$\frac{m_0^2}{g_0^2} \equiv \frac{T-T_c}{\Delta T_G} \equiv \tau. \tag{94}$$

A doubly logarithmic plot of the inverse square coherence length $\xi^{-2} \propto m^2/g_0$ in Fig. 13 shows then the slope changes from the free-field value 1 at large temperatures to 2ν near the critical temperature T_c. Similar crossover plots exist for all observable quantities in this field theory.

Finally, we use the series (75) for $\bar{g}(x)$ to calculate the beta-function which plays an important role in the renormalization group approach to scaling (in contrast to our explicit theory). By definition, we obtain it as a function of x from the logarithmic derivative of $\bar{g}(\bar{g}_0)$ [compare (53)]

$$\beta(x) = -\bar{g}_0\frac{d\bar{g}}{d\bar{g}_0} = f(x)\,\bar{g}'(x), \tag{95}$$

the result being

$$\begin{aligned}\beta(x) = &-1.622136195638856x + 2.73912944193321x^2 - 1.676962833531292x^3 + 0.5230145612386834x^4\\ &+0.1405773254892622x^5 - 0.06197010583664305x^6 - 0.06200066522622774x^7.\end{aligned} \tag{96}$$

The convergence of this strong-coupling expansion is seen by going to the weak-coupling limit $x = 1$ where $\beta(1) = -0.0046$ rather than the exact value 0. Expressing x as a function of $\bar{g}$ via (78), we obtain

$$\begin{aligned}\beta(\bar{g}) = &\ 0.805\Delta\bar{g} + 0.4980812505494033\Delta\bar{g}^2 - 0.04513957559397346\Delta\bar{g}^3 - 0.002836436593862963\Delta\bar{g}^4\\ &+0.000811067947065738\Delta\bar{g}^5 + 0.002150487674009535\Delta\bar{g}^6 - 0.002024061592617085\Delta\bar{g}^7\end{aligned} \tag{97}$$

which is plotted in Fig. 14. From this we can derive the function $\omega(\bar{g})$ by a simple derivative with respect to $\bar{g}$.

$$\begin{aligned}\omega(\bar{g}) = &\ 0.805 + 0.996162501098807\Delta\bar{g} - 0.1354187267819204\Delta\bar{g}^2 - 0.01134574637545185\Delta\bar{g}^3\\ &+0.004055339735328691\Delta\bar{g}^4 + 0.01290292604405721\Delta\bar{g}^5 - 0.01416843114831959\Delta\bar{g}^6\end{aligned} \tag{98}$$

At $\bar{g} = \bar{g}^*$, it has the value 0.805, as it should. In the weak-coupling limit $g = 0$, it is equal to $-.984$, very close to the exact value -1.

As a check for the consistent accuracy of our expansion procedures we calculate ω once more from

$$\omega = -\frac{\bar{g}_0}{\beta(\bar{g})}\frac{d\beta(\bar{g})}{d\bar{g}_0} = f(x)\frac{x}{\beta(x)}\beta'(x). \tag{99}$$

After expressing x in terms of $\Delta\bar{g}$ we obtain a series with coefficients very close to those in (99), with only a slightly worse weak-coupling limit -1.08 rather than -1.

VII. CONCLUSION

Variational strong-coupling theory proves to be a powerful tool for deriving the correct strong-coupling behavior and critical exponents of ϕ^4-theories in three dimensions. In a forthcoming work we shall find even more accurate results by taking into account information on the large-order behavior of all perturbation expansions involved. This can be done following the strategy developed for simple quantum mechanical systems in Ref. [20].

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While this paper was in the publishing process with Phys. Rev. D, the author received a preprint by A. Pelissetto and E. Vicari (University of Pisa preprint IFUP-TH 52/97) on the renormalization in O(N) models via $1/N$ expansions containing interesting analyticity observations which should lead to futher improvements of the theory.

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TABLE I. Equations determining the coefficients b_n in the strong-coupling expansion from the functions $b_n(c_N^{-q})$ and their derivatives. For brevity, we have suppressed the argument c_N^{-q} in the entries.

n	b_n	$-\gamma_{n-1}$
2	$b_2 + \gamma_1 b_1' + \frac{1}{2}\gamma_1^2 b_0''$	b_1'/b_0''
3	$b_3 + \gamma_2 b_1' + \gamma_1 b_2' + \gamma_1\gamma_2 b_0'' + \frac{1}{2}\gamma_1^2 b_1'' + \frac{1}{6}\gamma_1^3 b_0^{(3)}$	$(b_2' + \gamma_1 b_1'' + \frac{1}{2}\gamma_1^2 b_0^{(3)})/b_0''$
4	$b_4 + \gamma_3 b_1' + \gamma_2 b_2' + \gamma_1 b_3' + (\frac{1}{2}\gamma_2^2 + \gamma_1\gamma_3)b_0''$	$(b_3' + \gamma_2 b_1'' + \gamma_1 b_2'' + \gamma_1\gamma_2 b_0^{(3)}$
	$+\gamma_1\gamma_2 b_1'' + \frac{1}{2}\gamma_1^2 b_2'' + \frac{1}{2}\gamma_1^2\gamma_2 b_0^{(3)} + \frac{1}{6}\gamma_1^3 b_1^{(3)} + \frac{1}{24}\gamma_1^4 b_0^{(4)}$	$+\frac{1}{2}\gamma_1^2 b_1^{(3)} + \frac{1}{6}\gamma_1^3 b_0^{(4)})/b_0''$

TABLE II. Our critical exponents in comparison with those obtained by Padé-Borel resummation as compiled in Ref. [12] (superscript a), as well as earlier results indicated by subscripts b–e. They refer to six-loops expansions in $D = 3$ dimensions (b $\in$ [11], c $\in$ [21]), or to five-loop expansions in $\epsilon = 4 - D$ (d$\in$ [22], e$\in$ [23]). For each of our results we give the sixth-order approximation in parentheses to show the amount of extrapolation.

n	g_c	$\gamma(\gamma_6)$	$\eta(\eta_6)$	$\nu(\nu_6)$	α	β	ω (ω_6)
0		1.168(1.159)	0.025(0.0206)	0.592(0.586)			0.810(0.773)
	1.402[a]	1.160[a]	0.034[a]	0.589[a]	0.231[a]	0.305[a]	
	1.421±0.004[b]	1.161±0.003[b]	0.026±0.026[b]	0.588±0.001[b]	0.236±0.004[b]	0.302±0.004[b]	0.794 ± 0.06[b]
	1.421 ± 0.008[c]	1.1615 ± 0.002[c]	0.027 ± 0.004[c]	0.5880±0.0015[c]		0.3020±0.0015[c]	0.80 ± 0.04[c]
		1.160 ± 0.004[e]	0.031 ± 0.003[e]	0.5885±0.0025[e]		0.3025±0.0025[e]	0.82 ± 0.04[e]
1		1.241(1.235)	0.030(0.0254)	0.630(0.627)			0.805(0.772)
	1.401[a]	1.239[a]	0.038[a]	0.631[a]	0.107[a]	0.327[a]	
	1.416±0.0015[b]	1.241±0.004[b]	0.031±0.011[b]	0.630±0.002[b]	0.110 ± 0.008[b]	0.324 ± 0.06[b]	0.788 ± 0.003[b]
	1.416±0.004[c]	1.2410±0.0020[c]	0.031±0.004[c]	0.6300±0.0015[c]		0.3250±0.0015[c]	0.79±0.03[c]
			0.035 ± 0.002[d]	0.628±0.001[d]			0.80 ± 0.02[d]
		1.1239 ± 0.004[e]	0.037 ± 0.003[e]	0.6305±0.0025[e]		0.3265±0.0025[e]	0.81 ± 0.04[e]
2		1.318(1.306)	0.032(0.0278)	0.670(0.665)			0.800(0.772)
	1.394[a]	1.315[a]	0.039[a]	0.670[a]	-0.010[a]	0.348[a]	
	1.406±0.005[b]	1.316±0.009[b]	0.032±0.015[b]	0.669±0.003[b]	-0.007±0.009[b]	0.346±0.009[b]	0.78 ± 0.01[b]
	1.406±0.004[c]	1.3160±0.0025[c]	0.033±0.004[c]	0.6690±0.0020[c]		0.3455±0.002[c]	0.78±0.025[c]
			0.037 ± 0.002[d]	0.665±0.001[d]			0.79 ± 0.02[d]
		1.315 ± 0.007[e]	0.040 ± 0.003[e]	0.671±0.005[e]		0.3485±0.0035[e]	0.80 ± 0.04[e]
3		1.387(1.372)	0.032(0.0288)	0.705(0.700)			0.797(0.776)
	1.383[a]	1.386[a]	0.038[a]	0.706[a]	-0.117[a]	0.366[a]	
	1.392±0.009[b]	1.390±0.01[b]	0.031±0.022[b]	0.705±0.005[b]	-0.115±0.015[b]	0.362[b]	0.78 ± 0.02[b]
	1.391 ±0.004[c]	1.386±0.004[c]	0.033±0.004[c]	0.705±0.003[c]		0.3645±0.0025[c]	0.78 ± 0.02[c]
			0.037 ± 0.002[d]	0.79±0.02[d]			0.79 ± 0.02[d]
		1.390 ± 0.010[e]	0.040 ± 0.003[e]	0.710±0.007[e]		0.368±0.004[e]	0.79 ± 0.04[e]
4		1.451(1.433)	0.031(0.0289)	0.737(0.732)			0.795(0.780)
	1.369[a]	1.449[a]	0.036[a]	0.738[a]	-0.213[a]	0.382[a]	
5		1.511(1.487)	0.0295(0.0283)	0.767(0.760)			0.795(0.785)
	1.353[a]	1.506[a]	0.034[a]	0.766[a]	-0.297[a]	0.396[a]	
6		1.558(1.535)	0.0276(0.0273)	0.790(0.785)			0.797(0.792)
	1.336[a]	1.556[a]	0.031[a]	0.790[a]	-0.370[a]	0.407[a]	
7		1.599(1.577)	0.0262(0.0260)	0.810(0.807)			0.802(0.800)
	1.319[a]	1.599[a]	0.029[a]	0.811[a]	-0.434[a]	0.417[a]	
8		1.638(1.612)	0.0247(0.0246)	0.829(0.825)			0.810(0.808)
	1.303[a]	1.637[a]	0.027[a]	0.830[a]	-0.489[a]	0.426[a]	
9		1.680(1.643)	0.0233(0.0233)	0.850(0.841)			0.817(0.815)
	1.288[a]	1.669[a]	0.025[a]	0.845[a]	-0.536[a]	0.433[a]	
10		1.713(1.670)	0.0216(0.0220)	0.866(0.854)			0.824(0.822)
	1.274[a]	1.697[a]	0.024[a]	0.859[a]	-0.576[a]	0.440[a]	
12		1.763(1.716)	0.0190(0.0198)	0.890(0.877)			0.838(0.835)
	1.248[a]	1.743[a]	0.021[a]	0.881[a]	-0.643[a]	0.450[a]	
14		1.795(1.750)	0.0169(0.0178)	0.905(0.894)			0.851(0.849)
	1.226[a]	1.779[a]	0.019[a]	0.898[a]	-0.693[a]	0.457[a]	
16		1.822(1.779)	0.0152(0.0161)	0.918(0.907)			0.862(0.860)
	1.207[a]	1.807[a]	0.017[a]	0.911[a]	-0.732[a]	0.463[a]	
18		1.845(1.803)	0.0148(0.0137)	0.929(0.918)			0.873(0.869)
	1.191[a]	1.829[a]	0.015[a]	0.921[a]	-0.764[a]	0.468[a]	
20		1.864(1.822)	0.0125(0.0135)	0.938(0.927)			0.883(0.878)
	1.177[a]	1.847[a]	0.014[a]	0.930[a]	-0.789[a]	0.471[a]	
24		1.890(1.850)	0.0106(0.0116)	0.950(0.939)			0.900(0.894)
	1.154[a]	1.874[a]	0.012[a]	0.942[a]	-0.827[a]	0.477[a]	
28		1.909(1.871)	0.009232(0.01010)	0.959(0.949)			0.913(0.906)
	1.136[a]	1.893[a]	0.010[a]	0.951[a]	-0.854[a]	0.481[a]	
32		1.920(1.887)	0.00814(0.00895)	0.964(0.955)			0.924(0.915)
	1.122[a]	1.908[a]	0.009[a]	0.958[a]	-0.875[a]	0.483[a]	

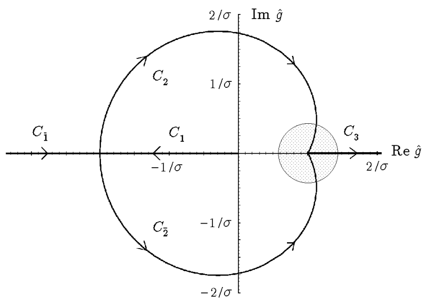

FIG. 1. Image of the left-hand cut in the complex $\hat{g}_0$-plane for $q = 3$.

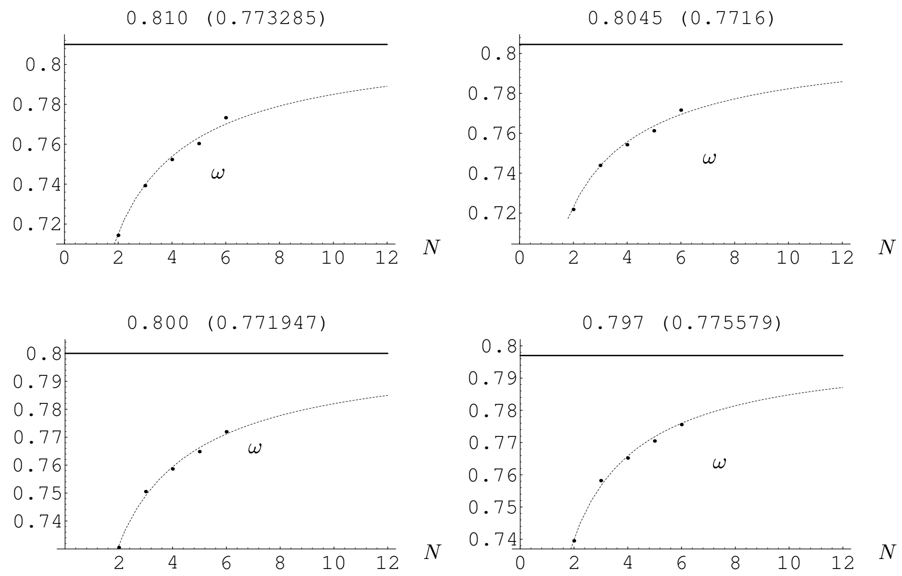

FIG. 2. Behavior of the strong-coupling values $\omega = \omega(\infty)$ with increasing order N of the approximation, for O(n)-symmetric theories with $n = 0,\ 1,\ 2,\ 3,\ \dots$. They are the critical exponents observable in the approach to scaling of the second-order phase transitions in percolating systems, Ising magnets, superfluid Helium, and the classical Heisenberg model. The points are fitted by expressions $\omega - b\, e^{-cN^{1-\omega}}$ (dashed line), determining by extrapolation the limiting values indicated by the horizontal lines, which are plotted in the first of Figs. 3 and listed in Table II. The highest approximation ω_6 is indicated in top of the line in parentheses.

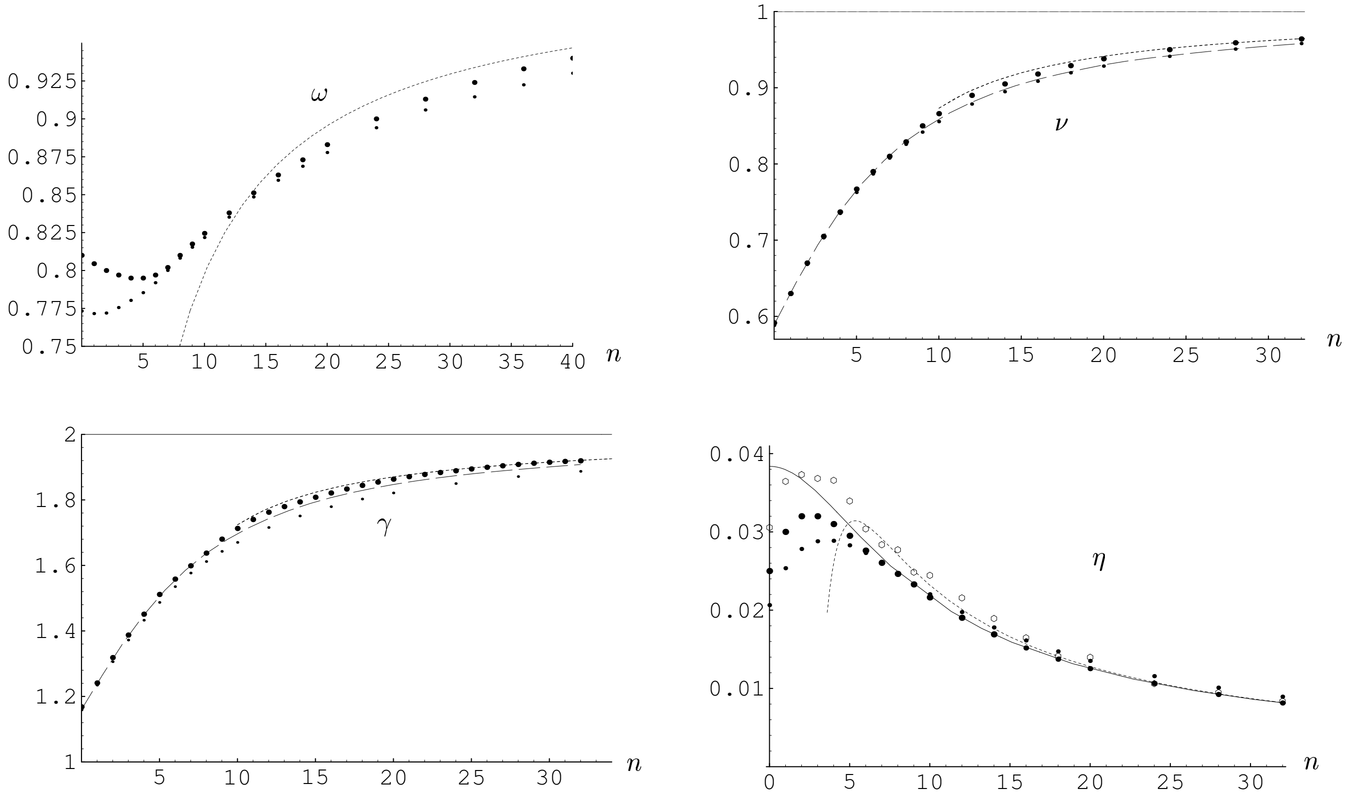

FIG. 3. Sixth-order approximations to $\omega, \nu, \gamma, \eta$ (thin dots), and their $N \to \infty$ -limit (fat dots). The dashed lines in the second and third figures are an interpolation to the Padé-Borel resummations of [12] (where ω was not calculated). Their data for η scatter too much to be represented in this way—they are indicated by small circles. The dotted curves show $1/n$ -expansions for all four quantities. Note that our results lie closer to these than those of Ref. [12]. The solid η-curve is explained after Eq. (71).

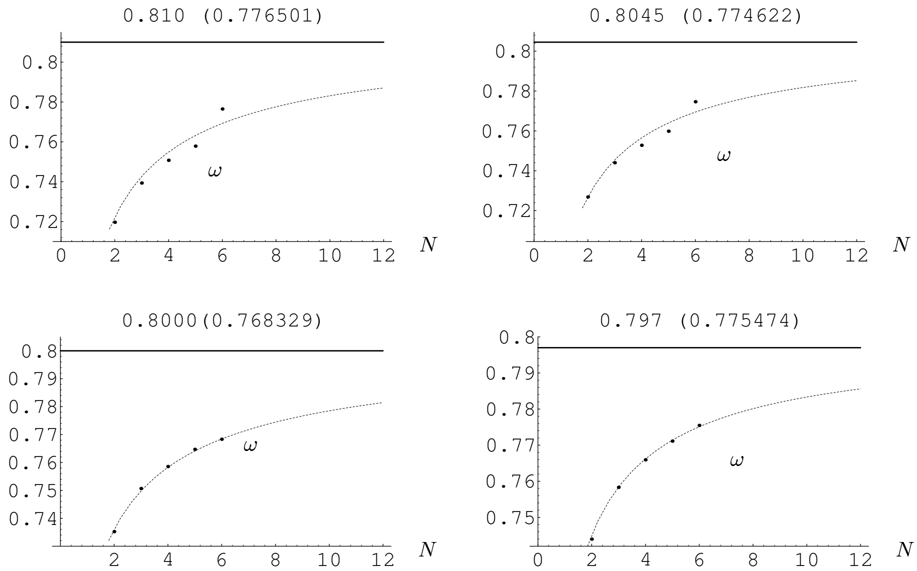

FIG. 4. Behavior of the self-consistent strong-coupling values $\omega = \omega(\infty)$ from Eq. (48) for $2/q = \omega$ with increasing order N of the approximation, for O(n)-symmetric theories with $n = 0,\ 1,\ 2,\ 3,\ \ldots$. The points are fitted by expressions $\omega - be^{-cN^{1-\omega}}$ (dashed line) with the same limiting values as in Fig. 2.

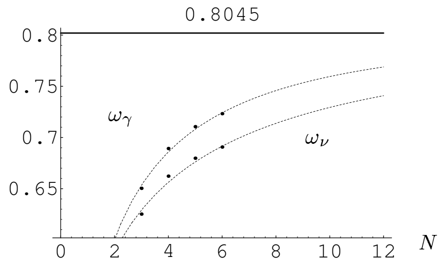

FIG. 5. The ω-exponents obtained from the expansions of γ and ν, but their convergence is much slower.

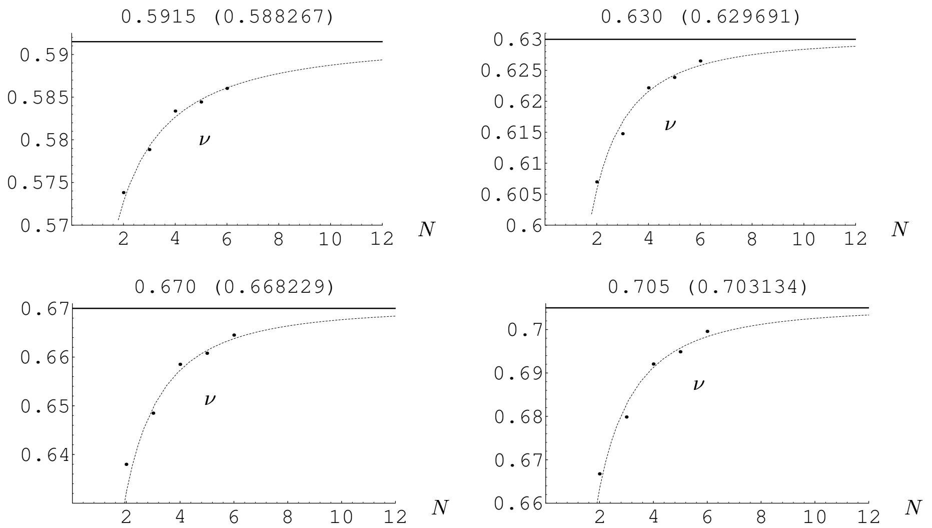

FIG. 6. Plot analogous to Fig. 2 for the critical exponent $\nu = \nu(\infty)$. The results of the extrapolation are plotted against n in the second of Figs. 3 and listed in Table II.

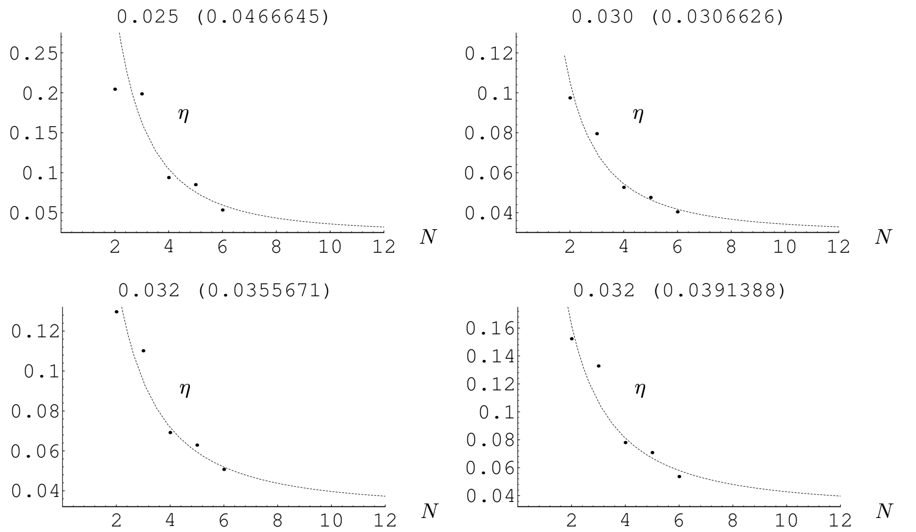

FIG. 7. Plot analogous to Figs. 2 for the critical exponent $\eta = \eta(\infty)$, illustrating the extrapolation procedure to $N \to \infty$ for $n = 0,\ 1,\ 2,\ 3$. The results of the extrapolation are plotted against n in the fourth of Figs. 3 and listed in Table II.

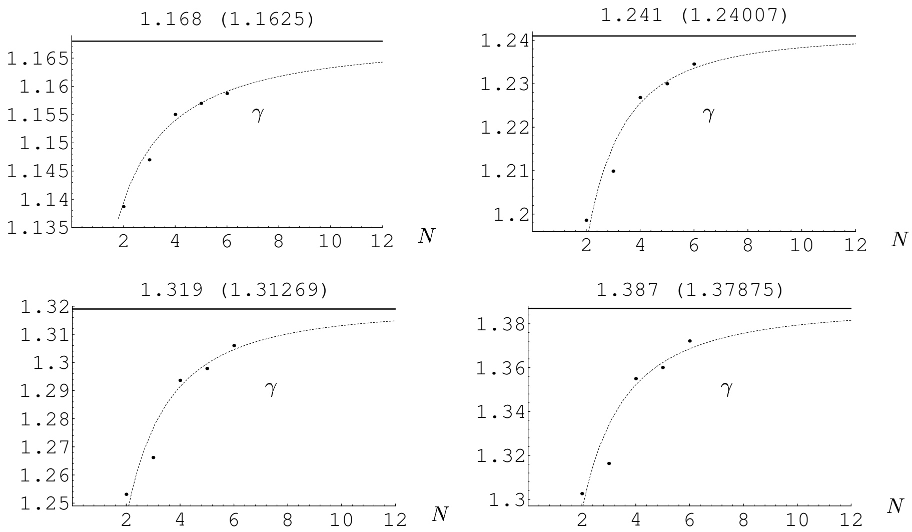

FIG. 8. Plot analogous to Figs. 2 for the critical exponent $\gamma = \gamma(\infty)$, illustrating the extrapolation procedure to $N \to \infty$ for $n = 0,\ 1,\ 2,\ 3$. The results of the extrapolation are plotted against n in the third of Figs. 3 and listed in Table II.

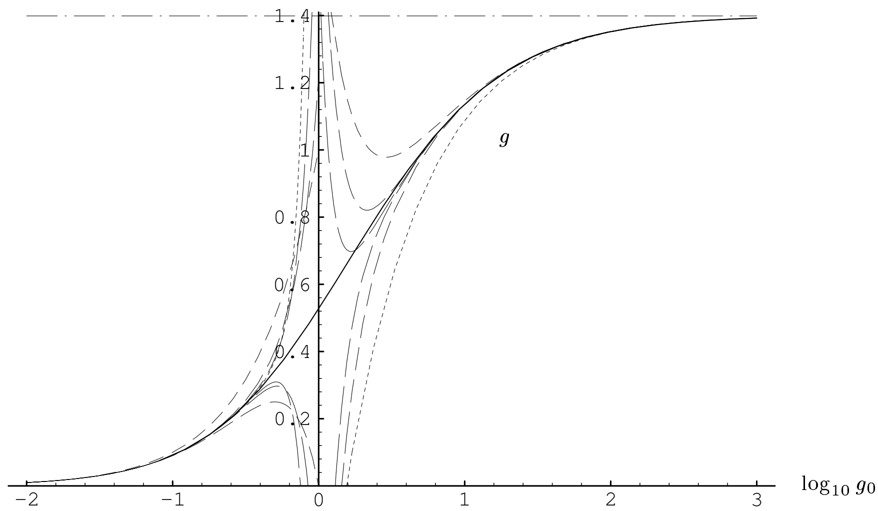

FIG. 9. Logarithmic plot of the renormalized coupling constant $\bar{g}$ as a function of the bare coupling $\bar{g}_0$ for all coupling strenghts. The small-$\bar{g}_0$ regimes shows the successive divergent perturbation expansions, the right-hand side the convergent strong-coupling expansions.

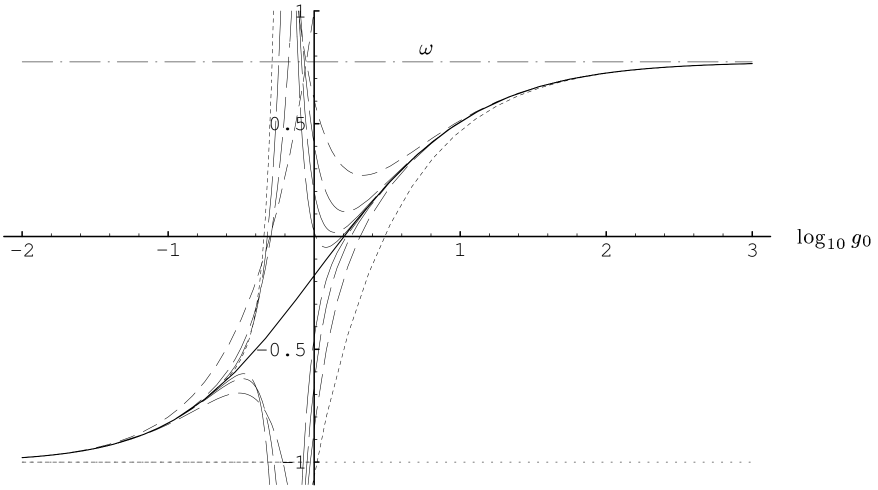

FIG. 10. Logarithmic plot of the full function $\omega(\bar{g}_0)$ for all coupling strenghts. The small-$\bar{g}_0$ regimes shows the successive divergent perturbation expansions, the right-hand side the convergent strong-coupling expansions. Increasing orders are indicated by increasing dash-lenghts.

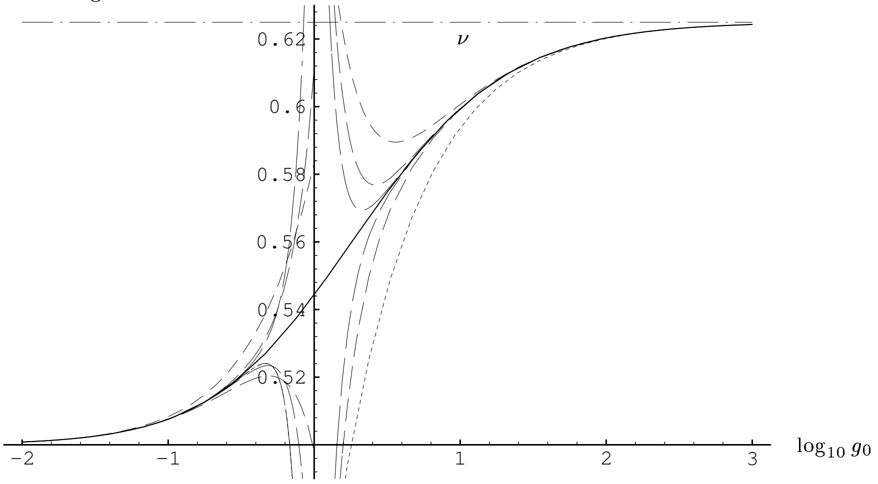

FIG. 11. Logarithmic plot of the full function $\eta_m(\bar{g}_0)$ for all coupling strenghts. The small-$\bar{g}_0$ regimes shows the successive divergent perturbation expansions, the right-hand side the convergent strong-coupling expansions.

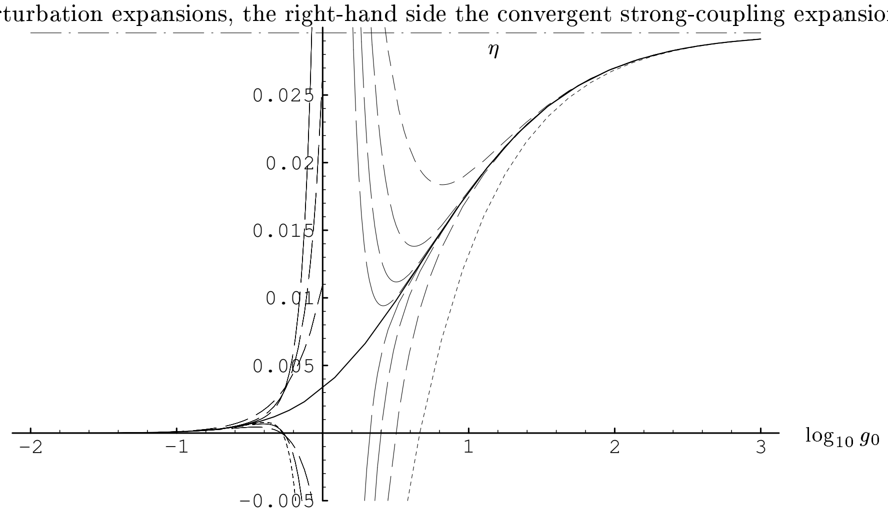

FIG. 12. Logarithmic plot of the full function $\eta(\bar{g}_0)$ for all coupling strenghts. The small-$\bar{g}_0$ regimes shows the successive divergent perturbation expansions, the right-hand side the convergent strong-coupling expansions.

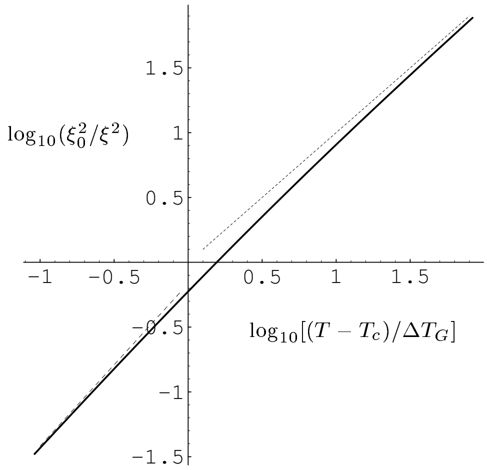

FIG. 13. Doubly logarithmic plot of the inverse square coherence length ξ^{-2} in arbitrary units against the reduced temperature $\tau \equiv (T - T_c)/\Delta T_G$, where ΔT_G is the Ginzburg temperature interval where the theory crosses over from free field to critical behavior. The dotted line shows the free-field limit with unit slope, the dashed line the strong-coupling limit with slope $2\nu \approx 1.252$.

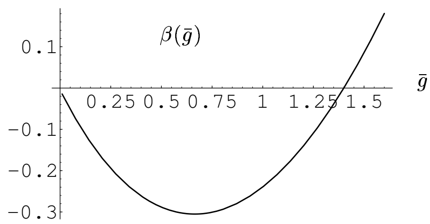

FIG. 14. Plot of the convergent strong-coupling expansion (97) for the beta function $\beta(\bar{g})$. The slope at the zero is the critical exponent $\omega = 0.805$. Note that the function converges well also at weak couplings. The curve isses the coordinate origin only by a very small amount.